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Cosmological models in Rosen's theory of gravitation

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Abstract. Solutions of Rosen's equations of gravitation for a homogeneous isotropic universe are obtained. The solutions resemble the Lemaitre universe A universal repulsive force, introduced in general relativity by a positive cosmological constant, appears here directly as a consequence of the (unmodified) field equations.

1. Introduction

Recently Rosen (1974) presented a bimetric theory of gravitation. As the theory agrees with observations (and general relativity) for weak gravitational fields, its viability has to be examined in situations where the weak-field approximation does not hold.

The following discussion shows that the cosmological solutions of the theory are compatible with the basic observational data. They also point to the existence of a repulsive force as a property of matter, in contrast to general relativity, where the repulsive force is introduced by means of a positive cosmological constant, as in the Lemaitre universe. Lemaitre models, which provide more time than Friedmann models for the formation of galaxies, have also been considered as an explanation of the cut-off of quasar red shifts at about the value of 2 (Shklovsky 1967, Brecher and Silk 1969, Burbidge and Burbidge 1971). At the same time, models with an arbitrarily chosen, non-vanishing cosmological constant are often treated with a certain reluctance (Ellis 1971, Burbidge and Burbidge 1971).

2. Solution of Rosen's field equations

Rosen's field equations (with the gravitational constant and the velocity of light both equal to unity) are

$$N_{\mu\nu}-\tfrac{1}{2}g_{\mu\nu}N = -8\pi\kappa T_{\mu\nu} \tag{1}$$

where $g_{\mu\nu}$ is the Riemannian metric tensor, $T_{\mu\nu}$ is the energy–momentum tensor and

$$N_{\mu}^{\nu} = \tfrac{1}{2}(g^{\nu\lambda}g_{\lambda\mu|\alpha})_{|\beta}\gamma^{\alpha\beta}; \tag{2}$$

the vertical stroke denotes a covariant derivative with respect to a flat-space metric tensor $\gamma_{\mu\nu}$, $\gamma^{\alpha\beta}$ is the inverse of $\gamma_{\mu\nu}$ and $\kappa = (G/\Gamma)^{1/2}$, G and Γ being determinants of $g_{\mu\nu}$ and $\gamma_{\mu\nu}$ respectively.

Let us assume a homogeneous isotropic universe filled with a perfect fluid, so that in co-moving coordinates $T_0^0 = \rho$, $T_1^1 = T_2^2 = T_3^3 = -p$, and the metric tensors in the spherical polar coordinates $(t, r, \theta, \Phi) = (x^0, x^6, x^2, x^3)$ given by

$$\gamma_{\mu\nu}\,dx^\mu\,dx^\nu = dt^2 - d\sigma^2, \qquad d\sigma^2 = dr^2 + r^2\,d\theta^2 + r^2 \sin^2\theta\,d\Phi^2$$
$$g_{\mu\nu}\,dx^\mu\,dx^\nu = \exp(2\varphi)\,dt^2 - \exp(2\psi)\,d\sigma^2. \tag{3}$$

The field equations (1) give

$$\varphi_{tt} - \varphi_{rr} - 2\varphi_r/r = -4\pi\kappa(\rho + 3p) \tag{4a}$$

$$\psi_{tt} - \psi_{rr} - 2\psi_r/r = 4\pi\kappa(\rho - p) \tag{4b}$$

where the subscripts t and r denote the respective partial derivatives.

The field equations can be solved analytically for a γ-law equation of state $p = (\gamma - 1)\rho$, where γ is a constant $(1 \leqslant \gamma \leqslant 2)$. The vanishing covariant divergence (with respect to $g_{\mu\nu}$) of $T_{\mu\nu}$ implies $\varphi_r = 0$ (since $p_r = 0$), and

$$\rho = L \exp(-3\gamma\psi)$$

where L is independent of t. Furthermore, $\rho_r = \varphi_r = 0$ implies $L_r = \psi_r = 0$, so that L is a (positive) constant and both φ and ψ are independent of r. The universe is infinite and the t = constant space sections have zero curvature.

One sees at once from (4b) that ρ enhances expansion, while p has the opposite effect. Indeed, if we rewrite (4b) in terms of the proper time τ defined by $d\tau^2 = dt^2 \exp(2\varphi)$ and compare it with Raychaudhuri's equation for the volume expansion (eg Ellis 1971) $3\dot{\psi}$ $(= 3\,d\psi/d\tau)$, we see that the term $(p - \rho)\exp(3\psi - \varphi)$ corresponds to the mass density in Newtonian cosmology, or $\rho + 3p$ in general relativity.

Let us choose the time origin at the maximum of $\zeta = \varphi - 3(\gamma - 1)\psi$ and define the time unit by $2\pi LA \exp(\zeta(0)) = 1$, where $A = -3\gamma^2 + 12\gamma - 8$. The solution of (4) is given by

$$\varphi = \varphi(0) + C(\lambda t - \ln\cosh t)$$
$$\psi = \psi(0) + C'(\lambda' t + \ln\cosh t)$$

where λ' is an arbitrary constant and

$$\lambda = 3\lambda'(\gamma - 1)(2 - \gamma)(3\gamma - 2)^{-1},$$
$$C = 2(3\gamma - 2)/A, \qquad C' = 2(2 - \gamma)/A.$$

We can group all terminal states ($|t| \to \infty$) into four classes, depending on the values of λ' and λ:

S1 : $\rho = 0$, τ finite

S2 : ρ infinite, τ finite

S3 : $\rho = 0$, τ infinite

S4 : ρ finite, τ finite.

The following seven model processes are allowed:

I. $\lambda \geqslant 1$: S2 $\to$ S3 (expansion)
II. $\lambda \leqslant -1$: S3 $\to$ S2 (contraction);

for $-1 < \lambda < 1$:

III. $\lambda' > 1$: S2 → S1 (expansion)
IV. $\lambda' = 1$: S4 → S1 (expansion)
V. $-1 < \lambda' < 1$: S1 → finite $\rho > 0$ → S1 (contraction followed by an expansion)
VI. $\lambda' = -1$: S1 → S4 (contraction)
VII. $\lambda' < -1$: S1 → S2 (contraction).

3. Discussion

If we assume the pressure to be exactly zero ($\gamma = 1$); only models III to VII are available ($\lambda = 0$). The same holds for $\gamma = 2$.

Since the real universe is expanding, we have a choice of cosmological models I, III, IV and V.

Model I starts with a big bang and expands for an unlimited time. The deceleration parameter $q = -(\ddot{\psi}+\dot{\psi}^2)\dot{\psi}^{-2}$ turns out to be bounded as follows: for γ very close to 1,

$$-1 < q(-\infty) < -\tfrac{1}{2}, \qquad -\tfrac{19}{18} < q(0) < -1, \qquad -\tfrac{3}{2} < q(\infty) < -1,$$

where the argument is t, and for $\gamma = \frac{4}{3}$,

$$0 < q(-\infty) < \tfrac{5}{4}, \qquad -\tfrac{2}{9} < q(0) < 0, \qquad -\tfrac{5}{8} < q(\infty) < 0.$$

The observational limits (Ellis 1971) $|q(\text{present time})| < 5$ are satisfied. Values of the Hubble constant $\dot{\psi}$ and the mass density ρ at present can be made to agree with the observed values by adjusting the remaining free parameters $L, \psi(0), \varphi(0)$ and also τ(present). The same holds for models III to V, but here $|q| < 5$ only for a limited range of τ.

Models III to V all have S1 ($\rho = 0$ at finite τ) as a final state and are thus geodesically incomplete in the future. Their qualitative properties can be illustrated by the solution for $\gamma = 1, \varphi(0) = 0$:

$$e^{\varphi} = 1-\tau^2; \qquad \dot{\psi} = 2(\lambda'+\tau)(1-\tau^2)^{-1}, \qquad \tau = \tanh t,$$
$$\rho \sim [(1-\tau)^{1+\lambda'}(1+\tau)^{1-\lambda'}]^3, \qquad q = -\tfrac{3}{2}-\tfrac{1}{2}(1-\lambda'^2)(\lambda'+\tau)^{-2}.$$

S1 as a final state is physically possible only for a perfectly homogeneous fluid, or for a fluid consisting of structureless particles, where the concept of time loses sense for $\rho \to 0$. If we assume that material bodies of some structure will persist in the future, the approximations of perfect homogeneity and the γ-law equation of state will ultimately fail as $\rho \to 0$; therefore S1 has to be rejected as a final state of the universe.

This leaves us with model I as the only possibly realistic one from those considered.

The γ-law equation of state with a constant γ is, however, a poor approximation when applied to the whole range of matter densities. It cannot be excluded that the field equations combined with a more realistic equation of state would admit other models, eg of type V: S1 → finite $\rho > 0$ → S3 (S1 seems admissible as an initial state). Such an analysis would probably require numerical solution and is beyond the scope of this paper.

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